NAG C Library Function Document

nag_dtrsyl (f08qhc)

1 Purpose

nag_dtrsyl (f08qhc) solves the real quasi-triangular Sylvester matrix equation.

2 Specification

```
void nag_dtrsyl (Nag_OrderType order, Nag_TransType trana, Nag_TransType tranb,
      Nag_SignType sign, Integer m, Integer n, const double a[], Integer pda,
      const double b[], Integer pdb, double c[], Integer pdc, double *scale,
      NagError *fail)
```

3 Description

nag_dtrsyl (f08qhc) solves the real Sylvester matrix equation

$$\mathrm{op}(A)X \pm X\mathrm{op}(B) = \alpha C,$$

where $\mathrm{op}(A) = A$ or A^T, and the matrices A and B are upper quasi-triangular matrices in canonical Schur form (as returned by nag_dhseqr (f08pec)); α is a scale factor (≤ 1) determined by the function to avoid overflow in X; A is m by m and B is n by n while the right-hand side matrix C and the solution matrix X are both m by n. The matrix X is obtained by a straightforward process of back substitution (see Golub and Van Loan (1996)).

Note that the equation has a unique solution if and only if $\alpha_i \pm \beta_j \neq 0$, where $\{\alpha_i\}$ and $\{\beta_j\}$ are the eigenvalues of A and B respectively and the sign ($+$ or $-$) is the same as that used in the equation to be solved.

4 References

Golub G H and Van Loan C F (1996) *Matrix Computations* (3rd Edition) Johns Hopkins University Press, Baltimore

Higham N J (1992) Perturbation theory and backward error for $AX - XB = C$ *Numerical Analysis Report* University of Manchester

5 Parameters

1: **order** – Nag_OrderType *Input*

On entry: the **order** parameter specifies the two-dimensional storage scheme being used, i.e., row-major ordering or column-major ordering. C language defined storage is specified by **order** = **Nag_RowMajor**. See Section 2.2.1.4 of the Essential Introduction for a more detailed explanation of the use of this parameter.

Constraint: **order** = **Nag_RowMajor** or **Nag_ColMajor**.

2: **trana** – Nag_TransType *Input*

On entry: specifies the option $\mathrm{op}(A)$ as follows:

if **trana** = **Nag_NoTrans**, then $\mathrm{op}(A) = A$;

if **trana** = **Nag_Trans** or **Nag_ConjTrans**, then $\mathrm{op}(A) = A^T$.

Constraint: **trana** = **Nag_NoTrans**, **Nag_Trans** or **Nag_ConjTrans**.

3: **tranb** – Nag_TransType *Input*

On entry: specifies the option $\text{op}(B)$ as follows:

if **tranb** = **Nag_NoTrans**, then $\text{op}(B) = B$;

if **tranb** = **Nag_Trans** or **Nag_ConjTrans**, then $\text{op}(B) = B^T$.

Constraint: **tranb** = **Nag_NoTrans**, **Nag_Trans** or **Nag_ConjTrans**.

4: **sign** – Nag_SignType *Input*

On entry: indicates the form of the Sylvester equation as follows:

if **sign** = **Nag_Plus**, then the equation is of the form $\text{op}(A)X + X\text{op}(B) = \alpha C$;

if **sign** = **Nag_Minus**, then the equation is of the form $\text{op}(A)X - X\text{op}(B) = \alpha C$.

Constraint: **sign** = **Nag_Plus** or **Nag_Minus**.

5: **m** – Integer *Input*

On entry: m, the order of the matrix A, and the number of rows in the matrices X and C.

Constraint: **m** ≥ 0.

6: **n** – Integer *Input*

On entry: n, the order of the matrix B, and the number of columns in the matrices X and C.

Constraint: **n** ≥ 0.

7: **a**$[dim]$ – const double *Input*

Note: the dimension, dim, of the array **a** must be at least $\max(1, \textbf{pda} \times \textbf{m})$.

If **order** = **Nag_ColMajor**, the (i, j)th element of the matrix A is stored in $\textbf{a}[(j-1) \times \textbf{pda} + i - 1]$ and if **order** = **Nag_RowMajor**, the (i, j)th element of the matrix A is stored in $\textbf{a}[(i-1) \times \textbf{pda} + j - 1]$.

On entry: the m by m upper quasi-triangular matrix A in canonical Schur form, as returned by nag_dhseqr (f08pec).

8: **pda** – Integer *Input*

On entry: the stride separating matrix row or column elements (depending on the value of **order**) in the array **a**.

Constraint: **pda** $\geq \max(1, \textbf{m})$.

9: **b**$[dim]$ – const double *Input*

Note: the dimension, dim, of the array **b** must be at least $\max(1, \textbf{pdb} \times \textbf{n})$.

If **order** = **Nag_ColMajor**, the (i, j)th element of the matrix B is stored in $\textbf{b}[(j-1) \times \textbf{pdb} + i - 1]$ and if **order** = **Nag_RowMajor**, the (i, j)th element of the matrix B is stored in $\textbf{b}[(i-1) \times \textbf{pdb} + j - 1]$.

On entry: the n by n upper quasi-triangular matrix B in canonical Schur form, as returned by nag_dhseqr (f08pec).

10: **pdb** – Integer *Input*

On entry: the stride separating matrix row or column elements (depending on the value of **order**) in the array **b**.

Constraint: **pdb** $\geq \max(1, \textbf{n})$.

11: **c**$[dim]$ – double *Input/Output*

Note: the dimension, dim, of the array **c** must be at least $\max(1, \textbf{pdc} \times \textbf{n})$ when **order** = **Nag_ColMajor** and at least $\max(1, \textbf{pdc} \times \textbf{m})$ when **order** = **Nag_RowMajor**.

If **order** $=$ **Nag_ColMajor**, the (i, j)th element of the matrix C is stored in **c**$[(j-1) \times$ **pdc** $+ i - 1]$ and if **order** $=$ **Nag_RowMajor**, the (i, j)th element of the matrix C is stored in **c**$[(i-1) \times$ **pdc** $+ j - 1]$.

On entry: the m by n right-hand side matrix C.

On exit: **c** is overwritten by the solution matrix X.

12: **pdc** – Integer *Input*

On entry: the stride separating matrix row or column elements (depending on the value of **order**) in the array **c**.

Constraints:

if **order** $=$ **Nag_ColMajor**, **pdc** $\geq \max(1, \mathbf{m})$;
if **order** $=$ **Nag_RowMajor**, **pdc** $\geq \max(1, \mathbf{n})$.

13: **scale** – double * *Output*

On exit: the value of the scale factor α.

14: **fail** – NagError * *Output*

The NAG error parameter (see the Essential Introduction).

6 Error Indicators and Warnings

NE_INT

On entry, **m** = $\langle$*value*$\rangle$.
Constraint: **m** ≥ 0.

On entry, **n** = $\langle$*value*$\rangle$.
Constraint: **n** ≥ 0.

On entry, **pda** $= \langle$*value*$\rangle$.
Constraint: **pda** > 0.

On entry, **pdb** $= \langle$*value*$\rangle$.
Constraint: **pdb** > 0.

On entry, **pdc** $= \langle$*value*$\rangle$.
Constraint: **pdc** > 0.

NE_INT_2

On entry, **pda** $= \langle$*value*$\rangle$, **m** $= \langle$*value*$\rangle$.
Constraint: **pda** $\geq \max(1, \mathbf{m})$.

On entry, **pdb** $= \langle$*value*$\rangle$, **n** $= \langle$*value*$\rangle$.
Constraint: **pdb** $\geq \max(1, \mathbf{n})$.

On entry, **pdc** $= \langle$*value*$\rangle$, **m** $= \langle$*value*$\rangle$.
Constraint: **pdc** $\geq \max(1, \mathbf{m})$.

On entry, **pdc** $= \langle$*value*$\rangle$, **n** $= \langle$*value*$\rangle$.
Constraint: **pdc** $\geq \max(1, \mathbf{n})$.

NE_PERTURBED

A and B have common or close eigenvalues, perturbed values of which were used to solve the equation.

NE_ALLOC_FAIL

Memory allocation failed.

NE_BAD_PARAM

On entry, parameter $\langle value \rangle$ had an illegal value.

NE_INTERNAL_ERROR

An internal error has occurred in this function. Check the function call and any array sizes. If the call is correct then please consult NAG for assistance.

7 Accuracy

Consider the equation $AX - XB = C$. (To apply the remarks to the equation $AX + XB = C$, simply replace B by $-B$.)

Let $\tilde{X}$ be the computed solution and R the residual matrix:

$$R = C - (A\tilde{X} - \tilde{X}B).$$

Then the residual is always small:

$$\|R\|_F = O(\epsilon)\ (\|A\|_F + \|B\|_F)\|\tilde{X}\|_F.$$

However, $\tilde{X}$ is **not** necessarily the exact solution of a slightly perturbed equation; in other words, the solution is not backwards stable.

For the forward error, the following bound holds:

$$\|\tilde{X} - X\|_F \le \frac{\|R\|_F}{sep(A,B)}$$

but this may be a considerable overestimate. See Golub and Van Loan (1996) for a definition of $sep(A,B)$, and Higham (1992) for further details.

These remarks also apply to the solution of a general Sylvester equation, as described in Section 8.

8 Further Comments

The total number of floating-point operations is approximately $mn(m+n)$.

To solve the **general** real Sylvester equation

$$AX \pm XB = C$$

where A and B are general nonsymmetric matrices, A and B must first be reduced to Schur form :

$$A = Q_1\tilde{A}Q_1^T \quad \text{and} \quad B = Q_2\tilde{B}Q_2^T$$

where $\tilde{A}$ and $\tilde{B}$ are upper quasi-triangular and Q_1 and Q_2 are orthogonal. The original equation may then be transformed to:

$$\tilde{A}\tilde{X} \pm \tilde{X}\tilde{B} = \tilde{C}$$

where $\tilde{X} = Q_1^T X Q_2$ and $\tilde{C} = Q_1^T C Q_2$. $\tilde{C}$ may be computed by matrix multiplication; nag_dtrsyl (f08qhc) may be used to solve the transformed equation; and the solution to the original equation can be obtained as $X = Q_1\tilde{X}Q_2^T$.

The complex analogue of this function is nag_ztrsyl (f08qvc).

9 Example

To solve the Sylvester equation $AX + XB = C$, where

$$A = \begin{pmatrix} 0.10 & 0.50 & 0.68 & -0.21 \\ -0.50 & 0.10 & -0.24 & 0.67 \\ 0.00 & 0.00 & 0.19 & -0.35 \\ 0.00 & 0.00 & 0.00 & -0.72 \end{pmatrix}, \quad B = \begin{pmatrix} -0.99 & -0.17 & 0.39 & 0.58 \\ 0.00 & 0.48 & -0.84 & -0.15 \\ 0.00 & 0.00 & 0.75 & 0.25 \\ 0.00 & 0.00 & -0.25 & 0.75 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 0.63 & -0.56 & 0.08 & -0.23 \\ -0.45 & -0.31 & 0.27 & 1.21 \\ 0.20 & -0.35 & 0.41 & 0.84 \\ 0.49 & -0.05 & -0.52 & -0.08 \end{pmatrix}.$$

9.1 Program Text

```
/* nag_dtrsyl (f08qhc) Example Program.
 *
 * Copyright 2001 Numerical Algorithms Group.
 *
 * Mark 7, 2001.
 */

#include <stdio.h>
#include <nag.h>
#include <nag_stdlib.h>
#include <nagf08.h>
#include <nagx04.h>

int main(void)
{
  /* Scalars */
  Integer  i, j, m, n, pda, pdb, pdc;
  Integer  exit_status=0;
  double   scale;
  NagError fail;
  Nag_OrderType order;
  /* Arrays */
  double *a=0, *b=0, *c=0;

#ifdef NAG_COLUMN_MAJOR
#define A(I,J) a[(J-1)*pda + I - 1]
#define B(I,J) b[(J-1)*pdb + I - 1]
#define C(I,J) c[(J-1)*pdc + I - 1]
  order = Nag_ColMajor;
#else
#define A(I,J) a[(I-1)*pda + J - 1]
#define B(I,J) b[(I-1)*pdb + J - 1]
#define C(I,J) c[(I-1)*pdc + J - 1]
  order = Nag_RowMajor;
#endif

  INIT_FAIL(fail);
  Vprintf("f08qhc Example Program Results\n\n");

  /* Skip heading in data file */
  Vscanf("%*[^\n] ");
  Vscanf("%ld%ld%*[^\n] ", &m, &n);
#ifdef NAG_COLUMN_MAJOR
  pda = m;
  pdb = n;
  pdc = m;
#else
  pda = m;
  pdb = n;
  pdc = n;
#endif

  /* Allocate memory */
  if ( !(a = NAG_ALLOC(m * m, double)) ||
       !(b = NAG_ALLOC(n * m, double)) ||
       !(c = NAG_ALLOC(m * n, double)) )
    {
      Vprintf("Allocation failure\n");
      exit_status = -1;
      goto END;
```

```
    }

  /* Read A, B and C from data file */
  for (i = 1; i <= m; ++i)
    {
      for (j = 1; j <= m; ++j)
        Vscanf("%lf", &A(i,j));
    }
  Vscanf("%*[^\n] ");
  for (i = 1; i <= n; ++i)
    {
      for (j = 1; j <= n; ++j)
        Vscanf("%lf", &B(i,j));
    }
  Vscanf("%*[^\n] ");
  for (i = 1; i <= m; ++i)
    {
      for (j = 1; j <= n; ++j)
        Vscanf("%lf", &C(i,j));
    }
  Vscanf("%*[^\n] ");

  /* Reorder the Schur factorization T */
  f08qhc(order, Nag_NoTrans, Nag_NoTrans, Nag_Plus, m, n, a, pda,
         b, pdb, c, pdc, &scale, &fail);
  if (fail.code != NE_NOERROR)
    {
      Vprintf("Error from f08qhc.\n%s\n", fail.message);
      exit_status = 1;
      goto END;
    }
  /* Print the solution matrix X stored in C */
  x04cac(order, Nag_GeneralMatrix, Nag_NonUnitDiag, m, n,
         c, pdc, "Solution matrix X", 0, &fail);
  if (fail.code != NE_NOERROR)
    {
      Vprintf("Error from x04cac.\n%s\n", fail.message);
      exit_status = 1;
      goto END;
    }
  Vprintf("\n SCALE = %10.2e\n", scale);
 END:
  if (a) NAG_FREE(a);
  if (b) NAG_FREE(b);
  if (c) NAG_FREE(c);

  return exit_status;
}
```

9.2 Program Data

```
f08qhc Example Program Data
  4  4                          :Values of M and N
  0.10   0.50   0.68  -0.21
 -0.50   0.10  -0.24   0.67
  0.00   0.00   0.19  -0.35
  0.00   0.00   0.00  -0.72    :End of matrix A
 -0.99  -0.17   0.39   0.58
  0.00   0.48  -0.84  -0.15
  0.00   0.00   0.75   0.25
  0.00   0.00  -0.25   0.75    :End of matrix B
  0.63  -0.56   0.08  -0.23
 -0.45  -0.31   0.27   1.21
  0.20  -0.35   0.41   0.84
  0.49  -0.05  -0.52  -0.08    :End of matrix C
```

9.3 Program Results

```
f08qhc Example Program Results

 Solution matrix X
          1         2         3         4
 1  -0.4209    0.1764    0.2438   -0.9577
 2   0.5600   -0.8337   -0.7221    0.5386
 3  -0.1246   -0.3392    0.6221    0.8691
 4  -0.2865    0.4113    0.5535    0.3174

 SCALE =   1.00e+00
```